

U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE

OFFICE NOTE 125

How to Avoid Complex Smoothing Operators

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Shuman (1957) pointed out that any symmetrical centered smoothing operator could be expressed as a sequence of "smoothing elements" that he defined as symmetrical centered three-point operators. For generality, however, smoothing elements must include those whose weights include complex numbers. For the large operator to be real, an element with a complex index, v , must be paired with one whose index is the complex conjugate of v . The index, v , he so defined that $(1-v)$ is the weight of the central point.

I will show in this paper that if the restriction of symmetry on smoothing elements is removed, complex weights may be avoided. Resolution of a five-point operator (in one dimension) into two three-point operators will serve this purpose.

The combined weights of a five-point operator may be developed from a sequence of two three-point operators with the following algorithm

$$\begin{vmatrix} a_0 \\ b_0 \\ c_0 \end{vmatrix} \times \begin{vmatrix} a_1 \\ b_1 \\ c_1 \end{vmatrix} = \begin{vmatrix} a_1 a_0 \\ a_1 b_0 + b_1 a_0 \\ a_1 c_0 + b_1 b_0 + c_1 a_0 \\ b_1 c_0 + c_1 b_0 \\ c_1 c_0 \end{vmatrix} = \begin{vmatrix} a \\ b \\ c \\ d \\ e \end{vmatrix}$$

All operators I am considering are "normal," that is, their weights add to unity. Therefore,

$$a_0 + b_0 + c_0 = 1 \quad (1a)$$

$$a_1 + b_1 + c_1 = 1 \quad (1b)$$

With these conditions on the three-point operators, the resultant five-point operator will be normal as well. Now, we restrict the five-point operator with the symmetry condition:

$$a = e = a_1 a_0 = c_1 c_0 \quad (2a)$$

$$b = d = a_1 b_0 + b_1 a_0 = b_1 c_0 + c_1 b_0 \quad (2b)$$

We further call the weight of its central point, c :

$$c = a_1 c_0 + b_1 b_0 + c_1 a_0$$

Now, I eliminate b_0 and b_1 from (2b), using (1):

$$a_1 + a_0 - 2 a_1 a_0 = c_1 + c_0 - 2 c_1 c_0$$

But because of (2a)

$$a_1 + a_0 = c_1 + c_0 \quad (3)$$

I now eliminate a_1 from (2a) using (3)

$$a_0^2 - (c_0 + c_1)a_0 + c_0 c_1 = 0$$

Solving this for a_0 , I find

$$a_0 = \frac{1}{2} [c_0 + c_1 \pm (c_0 - c_1)]$$

With (3), this gives two sets of conditions for the three-point operators:

$$\text{either } a_0 = c_0 \text{ and } a_1 = c_1 \quad (4a)$$

$$\text{or } a_0 = c_1 \text{ and } a_1 = c_0 \quad (4b)$$

The solution (4a) represents Shuman's symmetrical elements. Solution (4b) represents three-point asymmetrical operators which are mirror images of each other.

Now, I develop the five-point operator, using (4a)

$$\begin{vmatrix} a_0 \\ b_0 \\ a_0 \end{vmatrix} \times \begin{vmatrix} a_1 \\ b_1 \\ a_1 \end{vmatrix} = \begin{vmatrix} a_1 a_0 \\ a_1 b_0 + b_1 a_0 \\ a_1 a_0 + b_1 b_0 + a_1 a_0 \\ b_1 a_0 + a_1 b_0 \\ a_1 a_0 \end{vmatrix} = \begin{vmatrix} a \\ b \\ c \\ b \\ a \end{vmatrix} \quad (5a)$$

and also using (4b):

$$\begin{vmatrix} a_2 \\ b_2 \\ c_2 \end{vmatrix} \times \begin{vmatrix} c_2 \\ b_2 \\ a_2 \end{vmatrix} = \begin{vmatrix} c_2 a_2 \\ c_2 b_2 + b_2 a_2 \\ c_2 c_2 + b_2 b_2 + a_2 a_2 \\ b_2 c_2 + a_2 b_2 \\ a_2 c_2 \end{vmatrix} = \begin{vmatrix} a \\ b \\ c \\ b \\ a \end{vmatrix} \quad (5b)$$

Again, all operators are normal:

$$2 a_0 + b_0 = 1 \quad (6a)$$

$$2 a_1 + b_1 = 1 \quad (6b)$$

$$a_2 + b_2 + c_2 = 1 \quad (6c)$$

and the two five-point operators (5) are identical:

$$a = a_1 a_0 = c_2 a_2 \quad (7a)$$

$$b = a_1 b_0 + b_1 a_0 = b_2 (a_2 + c_2) \quad (7b)$$

$$c = 2 a_1 a_0 + b_1 b_0 = a_2^2 + b_2^2 + c_2^2 \quad (7c)$$

Any one of (7) may be derived from the other two and (6), however. I will use (7a) and (7b).

I eliminate a_2, c_2, b_0, b_1 from (7b) using (6):

$$b_2^2 - b_2 + a_0 + a_1 - 4 a_0 a_1 = 0$$

and solve for b_2 :

$$b_2 = \frac{1}{2} [1 \pm \sqrt{(1-4a_0)(1-4a_1)}] \quad (8)$$

Now, if a_0 and a_1 are complex conjugate, i. e., in terms of Shuman's smoothing indices, v :

$$v_0 = 2 a_0 = A + B i \quad (9a)$$

$$v_1 = 2 a_1 = A - B i \quad (9b)$$

the radicand in (8) is

$$\begin{aligned} (1-4a_0)(1-4a_1) &= (1-2v_0)(1-2v_1) = [(1-2A)-2Bi][(1-2A)+2Bi] \\ &= (1-2A)^2 + 4B^2 \geq 0 \end{aligned}$$

and therefore, b_2 is real.

I next eliminate c_2 from (7a) using (6c):

$$a_2^2 - (1-b_2)a_2 + a_1a_0 = 0$$

and solve for a_2 :

$$a_2 = \frac{1}{2} [1-b_2 \pm \sqrt{(1-b_2)^2 - 4a_0a_1}] \quad (10)$$

The radicand in (10), with substitutions for b_2, a_0, a_1 from (8) and (9) is

$$(1-b_2)^2 - 4a_0a_1 = \frac{1}{4} [1-2A \mp \sqrt{(1-2A)^2 + 4B^2}] \quad (11)$$

where choice of sign must be from the same position as in (8). Thus, (11) shows that the radicand of (10) is positive if the lower sign in (8) is chosen, and a_2 is then real. The dual sign in (10) is merely an indeterminacy in the order of the two mirror-image operators in (5b). Thus, one sign gives a_2 , the other c_2 .

In summary, if a five-point symmetrical smoothing operator is decomposed into two smoothing elements as in (5a), and if they are found to be complex, then two real asymmetrical three-point operators may be used instead as in (5b). Furthermore, the weights, a_2, b_2, c_2 may be calculated from the pair of complex conjugate numbers a_0 and a_1 by

$$b_2 = \frac{1}{2} [1 - \sqrt{1-4a_0}(1-4a_1)] \quad (12a)$$

$$a_2 = \frac{1}{2} [1-b_2 + \sqrt{(1-b_2)^2 - 4a_0a_1}] \quad (12b)$$

$$c_2 = \frac{1}{2} [1-b_2 - \sqrt{(1-b_2)^2 - 4a_0a_1}] \quad (12c)$$

Or they may be calculated from the real weights a and b taken from the five-point operator:

$$b_2 = \frac{1}{2} [1 - \sqrt{1-4b}]$$

$$a_2 = \frac{1}{2} [1-b_2 + \sqrt{(1-b_2)^2 - 4a}]$$

$$c_2 = \frac{1}{2} [1-b_2 - \sqrt{(1-b_2)^2 - 4a}]$$

As an example of how this works out, consider the set of smoothing elements devised by Shuman (1957) and used in NMC operations for 20 years.

$$v_0 = 2a_0 = -.22227 + .64240 i$$

$$v_1 = 2a_1 = -.22227 - .64240 i$$

$$v_2 = .49965$$

The last index, v_2 , is real, but the first two are complex conjugates. Before converting these to asymmetrical mirror-image operators, I take this opportunity to modify the set slightly, which will amount to little more than to make them "prettier." Expressing the first two in terms of A and B, as in (9), I note that A is very nearly $-2/9$, and I adopt $A = -2/9$. Similarly, v_2 is very nearly $1/2$, and I adopt $v_2 = 1/2$. Finally, I adopt $B = 0.642\ 915$. The table below shows the important characteristics of the two sets of operators, in terms of their responses, w , at various wave lengths (measured in units of grid increments).

	A = -.22227 B = .64240 $v_2 = .49965$		A = -2/9 B = .642 915 $v_2 = 1/2$	
	wave-length	w	wave-length	w
minimum w	11.512	1-.003 650	11.465	1-.003 705
w = 1	7.363	1.	7.318	1.
maximum w	5.705	1.003 868	5.711	1.003 705
w = 1	4.976	1.	4.992	1.
wave-length = 2	2.	.002 616	2.	0.

Equations (12) give for the new set:

$$a_2 = 1.383\ 298$$

$$b_2 = -.466\ 925$$

$$c_2 = .083\ 627$$

which may be written

$$\begin{aligned}
 & \begin{vmatrix} -1/9 + .321\,4575\,i \\ 11/9 - .642\,915\,i \\ -1/9 + .321\,4575\,i \end{vmatrix} \times \begin{vmatrix} -1/9 - .321\,4575\,i \\ 11/9 + .642\,915\,i \\ -1/9 - .321\,4575\,i \end{vmatrix} \\
 &= \begin{vmatrix} 1.383\,298 \\ -.466\,925 \\ .083\,627 \end{vmatrix} \times \begin{vmatrix} .083\,627 \\ -.466\,925 \\ 1.383\,298 \end{vmatrix} \\
 &= \begin{vmatrix} .115\,681 \\ -.684\,944 \\ 2.138\,526 \\ -.684\,944 \\ .115\,681 \end{vmatrix}
 \end{aligned}$$

Reference

Shuman, F. G., 1957: Numerical methods in weather prediction:
 II. Smoothing and filtering. Mon. Wea. Rev., 85, 357-361.



U.S. DEPARTMENT OF COMMERCE
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W345

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TO : Listed Below

FROM: *James E. McDonnell*
Chief, Meteorological Techniques Br/AD/NMC

SUBJ: New Data Format for "Bogus" Reports (Type 551) Office Note 124
in 'NWS.NMC.PROD.SFCBOG.TxxZ.LATEST'

In order to streamline the method of utilizing the NESS moisture estimates in our operations, an addition to Table SM.8a is being adopted. By storing the NESS moisture estimate (which is simply a number 1-10 at present), the procedures for utilizing them in the global analysis pre-processor (GLAPP) and the LFM moisture processor (LFMRH) can be greatly simplified. The necessary addition to O. N. 124 is given below:

TABLE SM.8a	
Code Figure	Specification
95	Moisture estimate by category LLNNN

Definitions

LL Level indicator: 97 = station level (surface)
NNN Category number (integer)

In order to implement the change it would be highly desirable for GLAPP (Rasch), LFMRH (Costello) and LISTSF2 (Fleming) to be able to accomodate both the current method and the proposed method equally. The target date for introduction of the new method is April 19, 1978.

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